HOMEWORK 2

Problem 1 Let $\{x\} = x - [x]$ denote the fractional part of $x \in \mathbb{R}$. If $m > 1$, $(a, m) = 1$ and b is any integer, prove that

$$\sum_{x} \left\{ \frac{ax + b}{m} \right\} = \frac{1}{2}(m - 1), \tag{1}$$

where x runs over a complete system of residues modulo m;

$$\sum_{\xi} \left\{ \frac{a\xi}{m} \right\} = \frac{1}{2}\varphi(m), \tag{2}$$

where ξ runs over a reduced system of residues modulo m.

Problem 2 Let $m_1, m_2, \ldots, m_k$ be pair-wise relatively prime integers greater than 1 and

$$m_1 m_2 \cdots m_k = M_1 m_1 = M_2 m_2 = \cdots = M_k m_k.$$

(a) Prove that if $x_1, x_2, \ldots, x_k$ run, respectively, through complete systems of residues to moduli $m_1, m_2, \ldots, m_k$, the integers $x_1 M_1 + x_2 M_2 + \cdots + x_k M_k$ run through a complete system of residues modulo $m_1 m_2 \cdots m_k$.

(b) Prove that if $\xi_1, \xi_2, \ldots, \xi_k$ run, respectively, through reduced systems of residues to moduli $m_1, m_2, \ldots, m_k$, the integers $\xi_1 M_1 + \xi_2 M_2 + \cdots + \xi_k M_k$ run through a reduced system of residues modulo $m_1 m_2 \cdots m_k$

Problem 3 (a) Find the remainder after division of $(12371^{56} + 34)^{28}$ by 111.

(b) Is $2^{1093} - 2$ divisible by 1093^2?

Problem 4 Let $m > 1$ and $(a, m) = 1$. Prove that the congruence

$$ax \equiv b \mod m$$

always has a solution $x \equiv ba^{\varphi(m)-1} \mod m$.

Problem 5 Prove the following converse to the Wilson's theorem: if $P > 1$ is such that

$$(P - 1)! + 1 \equiv 0 \mod P$$

than P is a prime.

Problem 6 Prove that a necessary and sufficient condition for a congruence

$$f(x) \equiv 0 \mod p, \quad f(x) = x^n + a_1 x^{n-1} + \cdots + a_n; \quad n \le p$$

to have n solutions is that all coefficients of the remainder of division $x^p - x$ by $f(x)$ are multiples of p.

Problem 7 Solve the following congruences:

(a) $256x \equiv 179 \bmod 337$.

(b) $1215x \equiv 560 \bmod 2755$.

Problem 8 Find a congruence of degree less than 5 equivalent to

$$3x^{14}+4x^{13}+3x^{12}+2x^{11}+x^9+2x^8+4x^7+x^6+3x^4+x^3+4x^2+2x \equiv 0 \text{ mod } 5.$$